Brownian Motion / Wiener Process

See Brown's Experiment on Brownian Motion

History

- First observed by a biologist **Robert Brown** in 1827 when he was studying sexual relations of plants – in particular he was observing particles in grains of pollen which were constantly in motion.

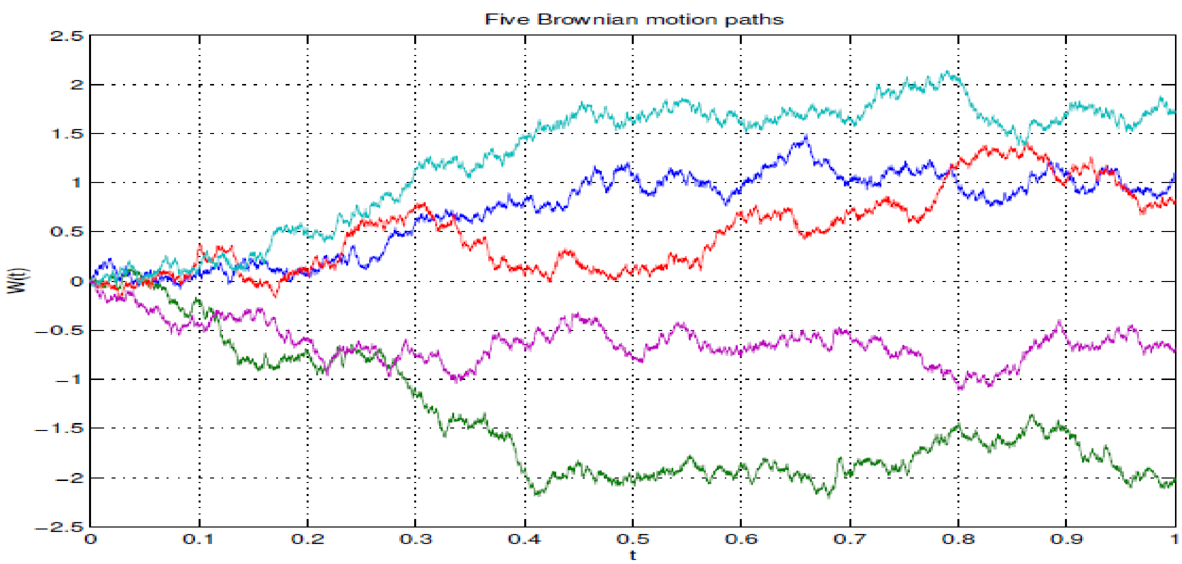
  See Brown's Experiment on Brownian Motion

- **Einstein's Theory** – The Osmosis Analogy (1905). He published a theoretical analysis of Brownian motion -- his focus was kinetic theory. Showed one can compute the actual size of a molecule based on a careful observation of Brownian motion.
- **Norbert Wiener** (1923) was the first to develop the mathematical theory of Brownian Motion.

Formal Definition

- $B = (B_t, t \in [0, \quad))$
- It is a gaussian stochastic process
- It starts at zero --- $B_0 = 0$
- It must be stationary
- It must have independent increments – (in finance these are called "shocks").
- Exhibits Markov property due to the independent increments
- For every $t > 0$, $B_t \sim N(0,t)$
- It has continuous sample paths – "no jumps" (unlike Poisson processes)
- Exhibits continuous time martingales. That is, the best prediction of the future net winnings per unit of stake in (s,t] is zero – that is, **the game is fair**. A stochastic process X_t is a martingale if $E[X_s \mid \mathcal{F}_t] = X_t$ for $s < t$

- The sample paths are extremely irregular – oscillate wildly – because the increments are independent resulting in **infinite total variation**.



- Brownian motion is 0.5-self-similar --- thus its sample path is nowhere differentiable. **This is why we need Ito calculus**.
- Note: self-similarity means that the properly scaled patterns of a sample path in any small or large interval have a similar shape, but they are not identical. **That is, the finer you look the more variation you see.**

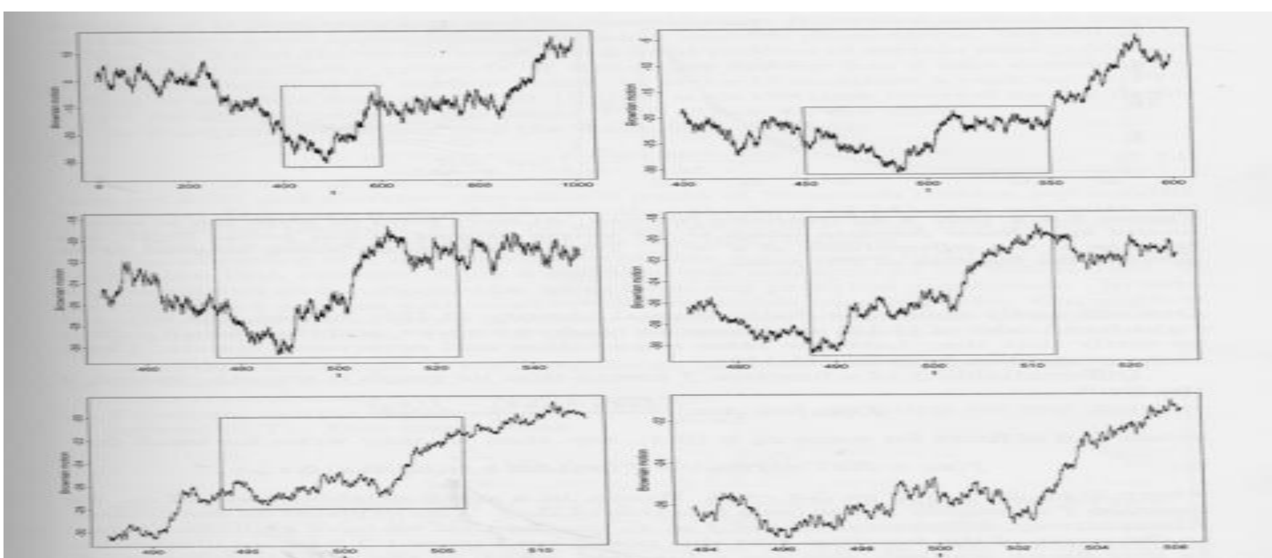


Figure 1.3.2 Self-similarity: *the same Brownian sample path on different scales. The shapes of the curves on different intervals look similar, but they are not simply scaled copies of each other.*

Consequences of Non-Differentiability

- **Dynamic Trading:** Some of the simplest dynamic trading strategies such as the Black-Scholes hedging, and Merton half stock / half cash trading, call for trades that are proportional to the change in the stock price. If however, the stock price is a **diffusion process** and there are transaction costs proportional to the size of the trade, then the **total transaction costs will either be infinite or very large**. This is a result of the **infinite total variation** of the Brownian motion.

- **Quadratic Variation:** For **continuous paths** (very small changes in t), the **increments should be small**. Quadratic variation is defined as the sum of the squares of these increments. In a Brownian motion, the quadratic variation terms are just small enough for the **sum not to go to zero or infinity** as $n \rightarrow \infty$. By Central Limit Theorem for a large n the sum should be close to its expected value… therefore the quadratic variation has a limit. Quadratic variation will be used in Ito calculus.

- **Trading Volatility**: The quadratic variation of a stock price is called it's "realized volatility". The fact that it is possible to buy and sell realized volatility says that the **Brownian motion model of stock price movement is not completely realistic**. That is, the model predicts that realized volatility is a constant, and therefore there is nothing to trade.

- **Almost sure convergence**: In continuous paths the limit may be infinite or the limit may be zero. **In continuous probability, there are many events that are impossible because they have a probability of zero, not because they do not exist**.