Stochastic Integral

Why Stochastic Integral?
- Brownian Motion are nowhere differentiable
- Brownian Motion has unbounded variation

Recall: Riemann Sum
\[ \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) \]
That is each term in the sum is the product of the value of the function at a given point and the length of the interval.

Recall: Riemann Integral
The Riemann Integral is the limit of the Riemann sums of a function. If the limit exists then the function is integrable.

Recall: Riemann-Stieltjes Integral
- Generalization of the Riemann Integral
- Allows one to compute the integral of function that is nowhere differentiable. Example: The path of a Brownian Motion.
- In essence the integral of a real valued function \( f \) of a real variable is stated with respect to a real function \( g \).
  - The RS exists when \( f \) and \( g \) do not have discontinuities at the same point
  - The function \( f \) has bounded p-variations and the function \( g \) has bounded q-variations for some \( p > 0 \) and \( q > 0 \) such that \( (1/p + 1/q) > 1 \)
- Still cannot be used for Brownian motion because of the unbounded variation of a Brownian sample path – therefore Itô’s Integral
Ito Stochastic Integral -- Motivation

- Integrals with respect to Brownian sample paths which cannot be defined in the RS sense, can hopefully be defined in the mean square sense. **Recall:** Brownian motion exhibit quadratic variation... that is it converges in the mean square sense -- or convergence in probability.

- A stochastic integral which is obtained as the mean square limit of RS sums, evaluated at the middle points of the intervals is called a Stratonovich integral which will be used to solve Ito stochastic differential equation. We need this integral because the classical chain rule of integration does not hold for Ito stochastic integrals.

So what is the Ito Stochastic Integral?

- Unlike RS Integral the Ito Stochastic Integral deals with random variables (stochastic processes)
- Integrates a non-differentiable function
- Allows one to integrate one stochastic process (the integrand) with respect to another stochastic process (the integrator). **Note:** The function to be integrated is the integrand. The integrator is a function that performs the integration.
- Generally, the integrator is the Brownian motion
- The result of the integration is another stochastic process.

How does it work?

- The random variable is defined as a limit of a certain sequence of random variables (convergence in probability)
- We choose a sequence of partitions of the interval from 0 to t.
- Then we construct Riemann sums -- this is where it gets tricky. What interval to use? Left, Right, middle?
- Typically we use the left end of the interval. For example, in finance this would imply we are deciding what to do and then observing a change in price.
• So visualize Ito’s Integral in Finance as:
  o The **integrand** is how much stock we hold
  o The **integrator** is the movement in price
  o The value of the integral is how much money we have in total at any given point in time

**Notation**

\[ Y_t = \int_0^t H_s \, dX_s \]

• Where: \( X \) is the Brownian motion (integrator); and \( H \) (integrand) is adapted (that is the value for \( H \) at any point on time \( t \) is only dependent on information available up until this time).

**Ito Stochastic Integral -- Properties Compared to Riemann and RS Integrals**

• Expectation = zero
• It is linear
• The process has continuous paths
• If the process is not simple, then the integral has one more property -- it exhibits the isometric property (that is, distance – preserving representation of one metric space as a subset of another).
**Ito’s Process**

**Recall: Wiener process (or Standard Brownian motion)**
- notation: $B_t$
- initial value: $B_0 = 0$
- small increments are independent and normal
- time points: $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$
- $\{\Delta B_i = B_{t_i} - B_{t_{i-1}}\}$ are independent
- $\Delta B_t = B_{t+\Delta t} - B_t \sim N(0, \Delta t)$.
- property: $B_t - B_0 \sim N(0, t)$
- **zero** drift and rate of variance change is 1. That is, $dx_t = 0 dt + 1 dB_t$

**Recall: Generalized Wiener process**
- $dx_t = \mu dt + \sigma dB_t$
- where the drift $\mu$ & rate of volatility change $\sigma^2$ are constant.

**Ito's process**
- $dx_t = \mu(x_t, t) dt + \sigma(x_t, t) dB_t$ -- stochastic diffusion equation
- where both drift and volatility are time-varying.
- Can also be written as in integral form as:
  - $x_t = x_0 + \int_0^t \mu(x_s, s) ds + \int_0^t \sigma(x_s, s) dB_s$

**Geometric Brownian motion**
- $dP_t = \mu P_t dt + \sigma P_t dB_t$
- such that $\mu(P_t, t) = \mu P_t$ and $\sigma(P_t, t) = \sigma P_t$ with $\mu$ and $\sigma$ are constant
- Therefore, using the Ito’s Process we have
  - $\mu(x_t, t) = \mu x_t$, and
  - $\sigma(x_t, t) = \sigma x$
Ito’s Lemma

Recall: Ito’s Integral

\[ Y_t = \int_0^t H_s dX_s \]

- **The problem:** Unless \( H \) is a simple process; we do not have a way to calculate the value of the Ito’s Integral and use it for simple operations. Therefore the Ito’s Lemma is needed.

- **Previously:** We concluded that we could define the integral in terms of a Brownian motion. That is,

\[ Y_t = \int_0^t B_s dB_s \text{ which converges in mean square to } \frac{1}{2} (B_t^2 - t) \]

Or the limit of the integral is now known.

Notice if the integral above did not have an embedded Brownian motion, the integral could be solved using the classic chain rule.

The value would be: \( \frac{1}{2} (B_t^2) \) ... which is similar to the limit – just offset by -0.5t

- This relationship therefore forces us to find a **correction factor** to the classical chain rule – which is the Ito’s Lemma.
Derivation of Ito’s Lemma

- Start with the classical deterministic Chain Rule of Differentiation
- Express the differential using the Taylor expansion
- Because the second and higher order terms are negligible for small changes in time, we ignore them.
- Then replace the functions f & g as a sample path exhibiting Brownian motion

Then for \( s < t \) and for a function that is **twice continuously differentiable** a simple version of the Ito’s Lemma can be stated as:

\[
f(B_t) - f(B_s) = \int_s^t f'(B_x) dB_x + \frac{1}{2} \int_s^t f''(B_x) dx
\]

**NOTE:**
- The first integral is Ito’s Stochastic Integral of \( f'(B) \)
- The second integral is the Riemann integral \( f''(B) \)

**Example 1: Function is a Power function**

Choose: \( f(t) = t^3 \). Then, \( f'(t) = 3t^2 \) and \( f''(t) = 6t \). Using Ito’s Lemma we can solve for \( s < t \) as follows:

\[
f(B_t) - f(B_s) = \int_s^t f'(B_x) dB_x + \frac{1}{2} \int_s^t f''(B_x) dx
\]

\[
\Rightarrow B_t^3 - B_s^3 = 3 \int_s^t B_x^2 dB_x + \frac{1}{2} \cdot 6 \int_s^t B_x dx
\]

\[
\Rightarrow B_t^3 - B_s^3 = 3 \int_s^t B_x^2 dB_x + 3 \int_s^t B_x dx
\]
Example 2: Function is an Exponential function

We have to re-derive the Ito’s Lemma and it is stated as:

\[
f(t, B_t) - f(s, B_s) = \int_s^t \left[ f_1(x, B_x) + \frac{1}{2} f_{22}(x, B_x) \right] dx + \int_s^t f_2(x, B_x) dB_x
\]

Choose: \( f(t, x) = e^{x - 0.5t} \).

Then,

\[
f_1(t, x) = \frac{\partial}{\partial t} f(t, x) = -\frac{1}{2} f(t, x)
\]

\[
f_2(t, x) = \frac{\partial}{\partial x} f(t, x) = f(t, x)
\]

\[
f_{22}(t, x) = \frac{\partial}{\partial x} \frac{\partial}{\partial x} f(t, x) = f(t, x)
\]

By substitution in the Ito’s Lemma we have:

\[
f(t, B_t) - f(s, B_s) = \int_s^t \left[ -\frac{1}{2} f(x, B_x) + \frac{1}{2} f(x, B_x) \right] dx + \int_s^t f(x, B_x) dB_x
\]

\[
f(t, B_t) - f(s, B_s) = \int_s^t [0] dx + \int_s^t f(x, B_x) dB_x
\]

\[
f(t, B_t) - f(s, B_s) = \int_s^t f(x, B_x) dB_x
\]
Example 3: Function is a Geometric Brownian Function

We have to re-derive the Ito’s Lemma and it is stated as:

\[ X_t - X_0 = \int_0^t \left[ f_1(x, X_s) + \frac{1}{2} f_{22}(x, X_s) \right] ds + \int_0^t f_2(x, X_s) dB_s \]

Choose: \( X_t = f(t, B_t) = e^{(c-0.5\sigma^2)t + \sigma B_t} \).

Then,

\[ f_1(t, x) = \frac{\partial}{\partial t} f(t, x) = c - \frac{1}{2} \sigma^2 f(t, x) \]

\[ f_2(t, x) = \frac{\partial}{\partial x} f(t, x) = \sigma f(t, x) \]

\[ f_{22}(t, x) = \frac{\partial}{\partial x} \frac{\partial}{\partial x} f(t, x) = \sigma^2 f(t, x) \]

By substitution in the Ito’s Lemma we have:

\[ X_t - X_0 = \int_s^t \left[ c - \frac{1}{2} \sigma^2 f(x, X_s) + \frac{1}{2} \sigma^2 f(x, X_s) \right] ds + \int_s^t \sigma f(x, X_s) dB_s \]

\[ X_t - X_0 = \int_0^t c ds + \int_0^t \sigma dB_s \]

\[ X_t - X_0 = c \int_0^t X_s ds + \sigma \int_0^t X_s dB_s \]
Stochastic Differential Equations

Definition – Deterministic Differential Equation

A differential equation is an equation that defines a relationship between a function and one or more derivatives of that function. Essentially, it provides a description of how something continuously changes over time. Some differential equations can have an analytical solution such that all future states can be known without simulation of the time evolution of the system. Some can only have a numerical solution with only a limited accuracy.

General notation: $dx(t) = a(t, x(t))dt, \ x(0) = x_o$
Definition – Stochastic Differential Equation

A stochastic differential equation is a deterministic differential equation which is perturbed by random noise.

Easiest way to add randomness is to randomize the initial condition. Then, \( x(t) \) becomes a stochastic process \( (X_t, t \in [0,T]) \)

\[
\frac{dX_t}{dt} = a(t, X_t) dt, \quad X_0(\omega) = Y(\omega)
\]

We often write the differential equation with an additional random noise term – say a Brownian motion.

\[
\frac{dX_t}{dt} = a(t, X_t) dt + b(t, X_t) dB_t, \quad X_0(\omega) = Y(\omega)
\]

Which we have seen can be written in integral form and called Ito’s Stochastic Differential Equation:

\[
X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s, \quad 0 \leq t \leq T
\]

- **NOTE:**
  - The first integral is Riemann integral
  - The second integral is the Ito’s Stochastic Integral
  - These equations can be solved as shown earlier using Ito’s Lemma