FIRM CAPITAL INVESTMENT DECISIONS

Lecture Objectives: a) to underscore the simplistic assumptions of certainty and perfect capital markets under which net present value (NPV) and related adjunct capital investment ranking criteria are derived and b) to incorporate uncertainty by the introduction of a real-option based approach to capital investment analytics.


1. Business firms are organized to produce goods or services.
2. We assume that management will act in the best interests of the firm's owners (stockholders).
3. For convenience, we will refer to the owners of the firm as stockholders. Further we will refer to the income provided to them as dividends.
4. There are two basic financial decisions confronting the firm in a world of certainty and perfect markets:
   a. what investments to undertake
   b. how should the investments be financed
5. The investment decision is comprised of two parts
   a. how much to invest and,
   b. the determination of the optimal investment portfolio.
6. The financing decision, which is not examined in detail in Managerial Econ., is to determine for a given amount of total investment, how much to finance through retained earnings as opposed to raising new money in the market (new stock or borrowing).
7. Since any income of the firm that is not retained must be paid to its owner, we shall refer to this financing problem as the dividend policy problem.

Where Do Capital Project Ideas Come From?

1. Created by the firm
2. For example – sales rep. May report that customers are looking for a particular type of variation to a product.
3. A firm’s growth, and its ability to remain competitive, depends upon a Cash Flow as related to existing and new products.
4. R&D department is responsible for generating new products which must be then folded into the manufacturing process.

The Importance of Capital Budgeting to the Firm

1. Effective capital budgeting can improve both the timing of asset acquisitions and the quality of assets purchased.
2. Asset acquisition, when planned properly, permits the firm to both acquire and install in an orderly manner.
3. When sales in a particular industry are increasing strongly, all firms tend to order capital goods at the same time; a situation that can lead to backlogs and undelivered capital items on a timely basis.
4. Asset expansion typically involves capital budgeting – this is the scale problem addressed under production and cost theory later – but with capital acquisition comes the problem of funds acquisition; hence, there is a need to address both working capital management and long terms sources of funds.

5. To the managerial economists, since the capital budgeting decision touches on:
   a. new / replacement assets (firm size or scale)
   b. cash flows (inflows and outflows - the P/L statement)
   c. working capital needs
   d. long term financing needs
   e. and, the incorporation of new technology (how the firm will combine labor, capital, and technology in the production of goods and services)

This managerial decision is viewed as one of (if not the most) the most important decisions faced by firm management.

The Focal Point: Cash Flow

Cash flows are not identical with profits or income. Changes in income can occur without any corresponding changes in cash flows.

The measurement of cash flow avoids some of the difficult problems underlying the measurement of corporate income that necessarily involve the accrual method of accounting. Typical questions we avoid:

1. In what time period should revenue be recognized?
2. What expenses should be treated as investments and therefore capitalized and depreciated over several time periods?
3. What method of depreciation should be used?
4. What costs may be inventoried?

Absolute vs. Relative (Incremental) Cash Flows

1. Absolute cash flows are those compared with zero cash flows; that is, where we assume no other cash flows exist.
2. Cash flows subtracted of one alternative subtracted from another alternative are referred to as relative or incremental cash flows.
3. It can be shown that the present value of relative cash flows will be the same as the difference between the present values of the absolute cash flows from two alternatives.
4. If an alternative accepted under absolute cash flows, we would compare, period-by-period, the actual cash flows with our previous forecasts. There is no similar identifiable series of cash flows that would be compared with relative cash flow estimates.
A Basic Cash Flow Equation

\[ C = (1 - t)(R - E) + tDEP_t \]

Where:

- \( C \) = the after-tax proceeds
- \( t \) = tax rate
- \( R \) = revenues
- \( E \) = expenses other than depreciation
- \( DEP_t \) = the depreciation expense taken for taxes
A Simple Model to Maximize Firm Value

We introduce a small four-variable cash flow model of the firm to exemplify the effects of certainty and perfect capital markets. Define a one-period model where decisions made today (the decision date) are shown with a naught subscript (0) and the end of the period decisions are made at time period one (1):

**The firm**

\[ X_0 - \text{income (revenues less expense)} \]
\[ X_1 - \text{income plus liquidating proceeds} \]
\[ I_0 - \text{the current investment budget} \]
\[ F_0 - \text{additional financing} \]
\[ V - \text{Value of the firm / Present value of the payments on currently outstanding shares.} \]

**The Individual / Investor**

\[ Y - \text{the individuals income from all sources other than his / her ownership interests in the firm.} \]
\[ VY - \text{Present value of Y, or present value of income external to the firm} \]
\[ s - \text{the fraction of the firm's shares owned by the individual} \]
\[ W - \text{Individuals current wealth} \]

\[ W = VY + sV \]

**NOTE:** it is the individual’s responsibility to maximize the value of \( VY \) from efforts in the workplace.

Maximization of Firm Value (V)

1. The question remains as to how the value of the firm is determined.

   It is the present value of the payments to the current stockholders.

2. Since there may be new shares issued in the future, we must be careful to distinguish between the total dividends paid by the firm and the amount that will be paid to current owners of currently outstanding shares (existing shares).

3. Individual wishes to \( \text{Max}(W) \), or choose firms which maximize \( V \).
How Is Value of V Determined?

In the multi-period case it is:

\[ V = D_0 + \sum_{t=1}^{T} \frac{D_t}{(1+i)^t} \]

But, as promised, let’s keep it simple – that is, let’s focus on the one period case.

\[ V = D_0 + \frac{D_1}{(1+i)} \]

given, \( X_0, I_0, X_1 \)

Define the firm by its cash flow function at time 0:

\[ X_0 + F_0 = D_0 + I_0 \]

Given cash flow, we can view the firm Financing Decision:

Given the income of the firm and its investment budget, paying an additional dollar of dividends requires it to raise another dollar in the market:

\[ (X_0 - D_0) + F_0 = I_0 \]

where \( (X_0 - D_0) \) is retained earnings.

An managerial question: How much \( I_0 \) should be financed with new money (\( F_0 \)) and how much with retained earnings?

Alternatively, we can view the firm’s dividend policy:

\[ D_0 = X_0 - I_0 + F_0 \]

The more money raised by \( F_0 \) the greater \( D_0 \) given income and investment.

Cash Flow At Time 1

\[ X_1 = D_1 + (1 + i)F_0 \]

We assume liquidation of the firm at time 1; that is, no new financing or investment at this time. Income from \( X_1 \) includes all proceeds from selling the firm’s assets. \( X_1 \) will be
distributed in part to the owners of the firm at time 0 (as D1) and in part to the new owners who supplied Fo.

New owners require payment of (1 + i)Fo, no more - no less (this is - certainty); hence, the dividends to the original owners is:

\[ D_1 = X_1 - (1 + i)F_0 \]

Let's substitute the D terms of (2) and (3) into (1) and simplify:

\[ V = X_0 - Io + \frac{X_1}{1+i} \]

Clearly, none of the variables are affected by the dividend-financing decision. Stated differently, the dividend decision does not affect the value of the firm. Or, as uncovered by your initial study of the M&M hypothesis we see again that under the market conditions assumed, the dividend decision is irrelevant to the determination of firm value.

Dissection: Cash Flow and Investment Policy

In order to uncover what effect, if any, current management has on the generation of firm cash flow (profitability), we separate cash flow into two components:

\[ X_1' - \text{the income the firm would earn if no investment took place (Io = 0) this includes time 1 liquidation. Under this scenario, firm management failed to undertake any new capital projects.} \]

\[ \Delta X_1 - \text{The effect on the net cash receipts (income) of the firm from any particular capital budget Io at time 1. This includes liquidating value.} \]

\[ X_1 = X_1' + \Delta X_1 \]

Let's substitute into (4)

\[ V = X_0 - Io + \frac{X_1' + \Delta X_1}{1+i} \]

or

\[ V = X_0 + \frac{X_1'}{1+i} + \frac{\Delta X_1}{1+i} - Io \]

but, both \( X_0 \) and \( X_1' \) are independent of the current investment decision. That is these cash flows are tied to prior year investment decisions. Therefore, to maximize the value of the firm, V, the firm financial manager should seek to maximize:

\[ \Delta V = \left[ \frac{\Delta X_1}{1+i} \right] - Io \]

This is the present value of the cash flow resulting from investing in \( Io \) no matter how the investment is financed.
Summary

1. Firm seeks to act in the best interests of the owners; therefore, the objective of the firm is to maximize the present value of the income stream provided to its current stockholders.
2. The pattern of the income distributed does not affect the firm's value -- dividend policy is irrelevant.
3. The objective of capital investment policy is also to maximize the value of the firm.
4. Maximizing the value of the firm is achieved by maximizing the present value of all incremental cash flows resulting from the capital investment process. However, in actual practice, to do so will likely involve the evaluation of opportunities in more than one period. In large-scale, complex capital investment decision-making we now know that:
   a. Cash flows should be measured on an incremental basis
   b. Cash flows should be measured on an after-tax basis.
   c. All indirect effects of the project throughout the firm should be included in the cash-flow calculations.
   d. Sunk costs should not be made considered when evaluating a project.
   e. The value of resources used in the project should be measured in terms of their opportunity costs.

5. Achieving Long-Run Performance – Traditional Approach
   a. The determination of the optimal capital budget at any single time cannot, in general, be considered independent of the budgets at other times.
   b. There is a need to review investment projects after they have been selected.
   c. The process of evaluating the actual cash flows as compared with the estimated cash flows requires the retention of some additional accounting information.
The Non-Complex Capital Investment Process

1. Generate alternative capital investment project proposals.

2. Project classification
   a. replacement -- maintain business / cost reduction
   b. expansion of existing products
   c. expansion of new products
   d. safety and environmental
   e. other

3. Estimate cash flows for the project proposals – (e.g., see a good MBA level finance or accounting textbook).

4. Evaluate and choose, from the alternatives available, those investments projects to implement.
   a. NPV
   b. IRR
   c. Payback / Discounted Payback
   d. Duration
   e. Uniform Annual Series
   f. Boulding’s Time Spread
   g. Other

Investment Project Classifications

In actual decision-making the capital investment process is not so easily implemented. Mangers must learn to cope with projects that are interrelated. For example, professional sports teams routinely evaluate players based on the their NPV. However, when the addition of one player (a quarterback) is dependent upon the availability of another player (a star wide receiver) the choices of what constitutes the optimal mix (budget) may not be so easy. Without the start wide receiver it may be questionable to land the high-priced quarterback. Of course, landing the quarterback may lead to the acquisition of another wide receiver. But, of course, there are budget constraints to consider!

In actual practice it will be necessary to assign proposed investment projects into one of three basic classifications: a) independent, mutually exclusive, and contingent (or dependent).

An independent project is defined as one whose acceptance or rejection does not necessarily eliminate other projects from consideration.

A mutually exclusive project is one whose acceptance precludes the acceptance of one or more alternative proposals.

A contingent project is one whose acceptance is dependent on the adoption of one or more other projects.
Quantitative Analysis

This section presents a review of basic ranking methods. Some of the listed methods should be reviewed at this level. These include: Payback (simple and discounted), Naïve rate-of-return, NPV, and IRR.

Simple Payback Period -- Payback(S)

The simple payback period is concerned with how long it takes to recover initial investment. Payback is defined as follows:

\[ \text{Payback} = \frac{C}{R} \]

where \( C \) is the cost of the project and \( R \) is the uniform revenue stream per period. Alternatively, when the revenue stream is not expected to be uniform, payback is calculated as follows:

\[ \text{PAYBACK} = P + \frac{C - \sum_{t=1}^{T} R_t}{R_{T+1}} \]

where:

\[ C - \sum_{t=1}^{T} R_t > 0.0, \text{ and } C - \sum_{t=1}^{T+1} R_t \leq 0.0 \]

Strengths of Payback Method

1. Widely used and easily understood.
2. Capital projects that return large early cash flows are favored by this technique.
3. Although it does not treat risk directly, it permits a financial manager to cope with risk by examining how long it will take to recoup initial investment.
4. Capital rationing issues are more easily addressed using the payback period.
5. The ease of use and interpretation permit decentralization of the capital budgeting decision. This enhances the chance of only worthwhile items reaching the final budget.

Weaknesses of the Payback Method

1. Cash flows beyond the payback period are ignored.
2. The time value of money is ignored.
3. It is difficult to distinguish between projects of different size when initial investment amounts are vastly divergent.
5. Overall payback periods are shortened by postponing replacement of depreciated plant and equipment; a policy which may do more harm than good to the production process.
Naive Rate of Return -- NROR

Whenever a decision maker uses a rate of return that does not rely upon time-value of money principles, this normally infers the use of a naive rate of return.

The naive rate of return, like payback methods, ignores the effects of cash flows beyond the payback period. In fact, the criticisms of payback are equally applicable to the naive rate of return. The naive rate of return is defined as:

\[ \text{NROR} = \frac{1}{\text{Payback}} \]

Discounted Payback Period -- Payback(D)

The discounted payback method is also known as the unrecovered investment analysis. The concept is similar to payback, but takes the time value of money into consideration:

\[ D = \text{URINV} = C(1 + k)^T - \sum_{t=1}^{T} R_t (1 + k)^{t-1} \]

Note a few special relationships. If \( k \) is an accurate measure of a firm's cost of capital, then the point at which \( \text{URINV} \) is equal to zero corresponds to the time at which the firm becomes indifferent to a project. That is, the firm is no better or worse off for having selected the project.

Internal Rate Of Return -- IRR (%)

The internal rate of return is defined as the reinvestment rate which, when compounded from period to period, equates the present value of cash inflows to the present value of cash outflows. Algebraically, the IRR is expressed as follows, where the \( r = \text{IRR} \) is the rate which causes the identity to hold:

\[ C_0 = \sum_{t=0}^{T} \frac{R_t}{(1 + r)^t} \]

The interested reader should note that as \( T \) approaches infinity, NROR and IRR are equal. Thus, under certain circumstances, the NROR is a good approximation of the IRR. These are the minimum conditions for NROR to approximate IRR:

1. The project life should be at least twice the payback period.
2. The cash flows must be nearly uniform.
3. The IRR is the discount rate that causes NPV to equal zero.
Net Present Value -- NPV

Net present value is the difference in the present value of cash inflows and outflows when discounted at the cost of capital. Algebraically, this is stated as:

\[ NPV = \sum_{t=1}^{T} \frac{R_t}{(1+k)^t} = C_0 \]

where \( C_0 \) is the present value of cash outflows if costs are incurred over a period of time.

Note that higher NPVs are more desirable than smaller NPVs. The specific decision rule for NPV is as follows:

- \( NPV \leq 0 \), reject project
- \( NPV > 0 \), accept project

Strengths of NPV versus IRR

- NPV is conceptually superior to payback and other accounting methods.
- NPV does not ignore any periods in the project life nor any cash flows.
- NPV takes into account the time value of money.
- NPV is easier to apply than IRR.
- NPV prefers early cash flows compared to later ones.

Weaknesses of NPV

- Unlike the calculation for IRR, the NPV calculation expects management to know the true cost of capital.
- NPV gives distorted comparisons between projects of unequal size or unequal economic life. To overcome this limitation, use NPV with the profitability index.
Profitability Index -- PI

The profitability index is the ratio of the present value of cash inflows to the present value of cash outflows.

\[ PI = \frac{PV \text{ of cash inflows}}{PV \text{ of cash outflows}} \]

A ratio of 1.0 or greater indicates that the project has an expected yield equal to or greater than the discount rate. Stated differently, the PI is a measure of a project's time weighted profitability per dollar of investment.

<table>
<thead>
<tr>
<th>NPV</th>
<th>PI</th>
<th>Expected Benefits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative</td>
<td>Less than 1</td>
<td>Less than required</td>
</tr>
<tr>
<td>Zero</td>
<td>Equal to 1</td>
<td>Deficient in Req. Ret.</td>
</tr>
<tr>
<td>Positive</td>
<td>Greater than 1</td>
<td>Req. Ret. exceeded</td>
</tr>
</tbody>
</table>

The profitability index is particularly useful when projects of unequal size must be compared. In this case the higher NPV project may appear more desirable simply because the associated cash flows are larger. The PI index eliminates this bias.
Level Annuity / Adjusted NPV

A problem with NPV and PI is the effect of unequal project lives on the calculations. A preferred way to handle this problem requires the calculation of level annuities.

The level annuities for two projects with unequal lives are calculated by following these steps:

1. Calculate the unadjusted NPV.
2. Calculate the capital recovery factor (CRF):
   \[
   CRF = \frac{k}{1-(1+k)^{-T}}, \text{ where } k = \text{cost of capital.}
   \]
3. Multiply NPV by CRF.
4. Convert the result obtained in point 3 to an adjusted NPV over the longest-lived project. That is, multiply NPV by CRF corresponding to the time period T .
5. The capital recovery factor (CRF) permits the decision-maker to view the cash flows as being evenly spread over a specified number of years. The formula shows that CRF is simply the reciprocal of the present value of annuity factor.
6. The adjusted NPVs (level annuities) obtained from this operation are more directly comparable across investment projects (as are the PIs).

Unrecovered Investment

This is a concept which is related to payback, but it takes the time value of money into account. The concept of unrecovered investment focuses attention on the nature of the capital recovery process.

Again, if we take \( C \) to be the cost of the investment at \( t = 0 \), and if \( k \) is the firm's per period opportunity cost of funds tied up in the project, then unrecovered investment is defined as:

\[
URINV = C(1 + k)^T - \sum_{t=1}^{T} R_t (1 + k)^{t-1}
\]

- It follows that the value of \( t \) for which \( URINV = 0 \) is the time-adjusted (discounted) payback period.
- If \( URINV \) is positive (greater than zero) then the project's time-adjusted cash flows do not cover the time-adjusted value of tying up funds over the expected life of the project.
- If the discount rate, \( k \), is set equal to the IRR then URINV will equal zero at time \( T \).
Frederick R. Macaulay's Duration

Duration is a very useful adjunct ranking measure for capital budgeting problems. Duration considers both the size and the timing of the cash flows. The duration measure can be interpreted as the average time that elapses for a dollar of present value to be received from the project. Alternatively, duration may be viewed as a weighted average of repayment times with weights equal to the present values of the cash flows at their respective dates. The duration calculation is expressed in units of time. For example, a calculation of 3.45 is expressed in years (time periods). A project with a shorter (lower) duration is preferred to a project with a longer (higher) duration calculation.

Two alternative computations of Duration are presented. One calculation is performed using the discount rate associated with the project's revenue stream. The other calculation uses the IRR as the discount rate. In this latter case where the discount rate (k) is set equal to the IRR, the denominator of the MacCaulay calculation reduces to the initial investment, C.

1. Given positive cash flows, duration will always be less than the expected life of the project.
2. There is generally a positive relationship between the project's expected life and duration. That is, a project with a longer expected life will have a higher duration than a project with a shorter expected life. But, the relationship is not direct because as the expected life increases the present value of the cash flows decline in value.
3. There is an inverse relationship between duration and cash flows. The higher the project's cash flow stream, the shorter the project's duration.

\[
\text{Duration} = D = \frac{\sum_{t=1}^{T} tR_t (1+k)^{-t}}{\sum_{t=1}^{T} R_t (1+k)^{-t}}
\]

NOTE: If the discount rate, k, is set equal to the IRR, then the denominator reduces to the initial investment, C.
Example

Given a fixed-rate investment with 10 years to maturity, a coupon rate of 8 percent on a face value of $100 and a current yield-to-maturity (YTM) of 10.42%, the following duration of 7.001 is derived accordingly:

<table>
<thead>
<tr>
<th>Col 1</th>
<th>Col 2</th>
<th>Col 3</th>
<th>Col 4</th>
</tr>
</thead>
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<tr>
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<td>Cash Flow (CF)</td>
<td>PV CF</td>
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<td></td>
<td></td>
</tr>
<tr>
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<td>7.245</td>
</tr>
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<td>8</td>
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</tr>
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<td>9</td>
<td>8</td>
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</tr>
<tr>
<td>10</td>
<td>108</td>
<td>40.082</td>
<td>400.817</td>
</tr>
</tbody>
</table>

**Price** 85.395  597.833

Duration (D) = 597.833 / 85.395 = 7.001

**Durations @ 10% yield for 6% and 8% Coupon**

<table>
<thead>
<tr>
<th>Coupon / Maturity (yrs)</th>
<th>6%</th>
<th>8%</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>4.41</td>
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<td>10</td>
<td>7.42</td>
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<tr>
<td>100</td>
<td>11.00</td>
<td>11.00</td>
</tr>
</tbody>
</table>

Still unsure about Duration? Click Here for a corresponding spreadsheet example.
Modified Duration \((D_{\text{mod}})\)

This is a simple adjustment to MacCauley's duration. Modified duration \(D_{\text{mod}}\) equals Macaulay duration divided by 1 plus the cost of capital \((k)\).

\[
D_{\text{mod}} = \frac{D}{1+k}
\]

Modified duration is important as it can be shown analytically that the value of a capital project will vary proportionally with modified duration for small changes in the cost of capital.

Modified duration is routinely used to estimate changes in the value of a capital project for a small change in the firm's discount rate. For example, the percentage change in the price (value) of an asset (project) given a change in interest rate (discount rate) levels can be estimated by applying modified duration:

\[
\% \Delta \text{Price (or the value of a project)} = -D_{\text{mod}} \times (\Delta k),
\]

where \(D_{\text{mod}}\) is MacCauley's modified duration and \(k\) is the firm's cost of capital (discount factor). But, the accuracy of the estimated change in value (price) deteriorates with larger changes in discount factors. Why? Because the modified duration calculation presented above is a linear approximation of an asset price change relationship that is known to follow a nonlinear (convex) function. This effect is referred to as convexity.

Generally speaking, an asset with a short expected life and high cash flows will have very low convexity (greater linearity in the relationship). This asset will also have a low Duration. In contrast, an asset with a long expected maturity and very low cash flows will have high convexity (and a high Duration).

In summary, a change in an asset's value (price) to the firm will depend on two factors: a) the asset's modified duration and b) its convexity. The relative effects of these two factors on the asset's value will depend on the size and timing of the associated cash flows and the magnitude of the change in cost of capital.

Estimating the Percentage Change in Bond Price

By way of example, given a traded bond when interest levels are expected to decrease by 75 basis points we can evoke modified duration to estimate the amount by which bond price will change (show variability). Assume a modified duration of 5.736:

\[
\%\Delta \text{Bond Price} = (-5.736) \times (-.75) = 4.302\%
\]

This approach assumes that there is a linearly declining relationship between bond price and interest rates.
Trading Strategies Using Duration:

If you expect a decline in interest rates, you should increase the average duration of your bond portfolio to experience maximum price volatility.

If you expect an increase in interest rates, you should reduce the average duration of your portfolio to minimize your price decline.

Immunization: for any interest-rate sensitive asset like a bank lending portfolio, obtaining protection from interest rate sensitivity is achieved by establishing a holding period that equals the duration of the assets. The duration of the portfolio is the durations weighted by the market value of the individual bonds in the portfolio.

Duration, Convexity and Bond Price

Ordinary bonds (not callable or convertible) have convexity.

Duration is a good measure of bond price volatility only when interest rates do not change by much over a short period of time. When this is not the case then convexity effects must be considered.

Price Approximation Using Modified Duration

The accuracy of the estimate of the price change deteriorates with larger changes in yields because modified duration is a linear approximation of the nonlinear bond price change function given large changes in interest rate.

- There is an inverse relationship between coupon and convexity (yield and maturity constant).
- There is a direct relationship between maturity and convexity (yield and coupon constant).
- There is an inverse relationship between maturity and convexity (coupon and maturity constant). This means that the price-yield curve is more convex at its lower-yield (upper left) segment.

For example: a short-term, high coupon bond, such as a 12 percent, 2-year bond, has very low convexity (it is almost a straight line). Conversely, a zero coupon, 30-year bond has high convexity.

**Summary:**

The change in a bond's price resulting from a change in yield can be attributed to:

- The bond's modified duration
- The bond's convexity

The effect of these two factors on the price change will depend on the characteristics of the bond (i.e., its convexity) and the size of the yield change.

**Effective Duration**

This topic is presented here as a definition. WinORS does not calculate effective duration.

Effective Duration is an alternative view (some may argue that it is an enhanced view) of duration. Because modified duration is based on Macaulay duration, it provides a reasonable approximation of an asset's change in value given small changes in the firm's overall cost of capital. Effective duration and convexity are measures of price sensitivity to **parallel yield changes**. They are calculated using approximate relationships for duration and convexity in terms of price changes due to yield changes. Prices for $\pm(\Delta Y)$ are calculated from a valid model that accounts for cash flow variability under different interest rate paths. Stated differently, effective duration and convexity measures are used to calculate instantaneous return to given yield change scenario.

Effective duration improves upon the standard MacCauley duration by measuring the expected change in value of a fixed income security or portfolio for a given change in interest rates. For example, if interest rates fell by one percent, the value of a security or portfolio having an effective duration of 2.0 generally would increase in price by two percent. An investment policy with a 5 year portfolio maturity limitation has an effective duration of approximately 4.6, indicating the portfolio's value would increase (or decrease) 4.6% with a one percent drop (rise) in interest rates. Effective duration is a superior method for defining the level of interest rate in portfolios compared to measuring its average final maturity or standard duration.

$$\text{Effective or Approximate Duration} = \frac{V_+ - V_-}{2V_0(\Delta Y)}$$
Where $V_-$ and $V_+$ represent estimated prices when yield decrease by $(\Delta Y)$ and increase by $(\Delta Y)$. $V_0$ represent initial observed price.

For securities with embedded options, if one has an accurate (valid) model of the price changes as a function of yield changes (e.g. binomial tree), one can use the approximate duration equation to calculate an effective duration measure or option-adjusted duration measure (OAD).

Effective duration measures the first order (linear) price/yield sensitivity. The linear portion of the price change of securities with embedded options due to interest changes can be approximated using effective duration:

Approximate Percentage Price Change (Linear) = - Effective Duration x Change in Yield

By contrast, modified duration assumes cash flows are unchanged when interest rates are changed. One can apply exact computation of modified duration to callable bonds assuming cash flows stay the same. However such a measure overestimates the price/yield sensitivity of callable bonds.

**Convexity**

*Convexity* measures the nonlinear behavior of price-yield relationship. Convexity assumes the cash flows are certain and do not vary with interest rate paths. However one can use approximate relationship for convexity based on an accurate price model to calculate price changes considering changes in cash flow patterns (optionally). Such a measure for securities with embedded options is referred to as effective convexity.

Effective or Approximate Convexity = \[\frac{V_- + V_+ - 2V_0}{V_0(\Delta Y)^2}\]

Approximate Percentage Price Change = - Effective Duration * $(\Delta Y)$ + (1/2) Effective Convexity * $(\Delta Y)^2$

Example: $V_0 = 100$, $V_-$ = 103 for +50bp, $V_+$ =95 for -50bp

Effective Duration = - $(95-103)/(2*100*0.005) = 8$ yrs
Effective Convexity = $(103+95-200)/(100*0.005^2) = 800$
Kenneth E. Boulding's Time Spread -- TS

The Boulding time spread (TS) measure reports the average time interval elapsing between sets of capital outlays and capital inflows.

TS is a measure of the average time between capital outlays and net cash receipts.

The formula to calculate either time spread is shown below. In the equation simply substitute the appropriate return as the value for $r$.

$$TS = \log \left[ \frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} R_t (1 + r)^{-t}} \right]$$

When used with the internal rate of return, TS reports on average how long the initial investment remains invested at the IRR.

Stated differently, TS is the point in time at which a single amount (the sum of the undiscounted cash inflows) is equal to the time-valued cash inflows over the life of the investment.

For example, if given an IRR of 21.406%, and TS is computed as 4.72 periods we could reason that the individual net cash flows could be replaced by a single cash flow of (sum of cash flows here) at 4.72 periods.
**Value at Risk: VaR**

VaR is an estimate of the level of loss on a project which is expected to equaled or exceeded with a given (small) probability.

**Pros**
- Represents risk in one number
- Measures downside risk (variance, by contrast, is two-sided)
- Applicable to nonlinear instruments

**Cons**
- May provide an inadequate view of risk
- May provide conflicting results at different confidence levels
- Non-convex and non sub-additive
- Difficult to use for non-normal distributions

**Alternative Definitions in the Literature**

- *VaR is a forecast of a given percentile, usually in the lower tail, of the distribution of returns on a portfolio over some period; similar in principle to an estimate of the expected return on a portfolio, which is a forecast of the 50th percentile.*
- *VaR is an estimate of the level of loss on a portfolio which is expected to be equaled or exceeded with a given, small probability.*
- "*VaR is the maximum loss over a target horizon such that there is a low, pre-specified probability that the actual loss will be larger."* Phillippe Jorion.
- "*VaR is a category of market risk measure that describes probabilistically the market risk of trading portfolios.*" Glyn Holton, *Contingency Analysis*.
- "*VaR is a dollar measure of the minimum loss that would be expected over a period of time with a given probability.*" Don Chance.
Random variable, $\xi$

Source: Prof. S. Uryasev, Univ of Florida

$$p(\xi)$$

Maximal value

$1 - \alpha$

Probability

$\text{VaR}$

Source: Prof. S. Uryasev, Univ of Florida
Goldman Sachs VaR Reaches Record on Risks Led by Equity Trading

By Christine Harper - July 14, 2009 20:01 EDT

July 15 (Bloomberg) -- Goldman Sachs Group Inc. ratcheted up risk-taking to an all-time high in the second quarter, increasing equity bets 58 percent to amass record trading revenue and quarterly earnings.

Value-at-risk, a measure of how much money the firm could lose in a day's trading, rose to $245 million from $240 million in the first quarter, the New York-based firm said yesterday. The figure stood at $184 million last May (see table, below). Most of the increase during the second quarter stemmed from equity-trading risk, which surged to an average of $60 million per day from $38 million in the previous three months.

Goldman Sachs’s move to become a bank holding company in September to win the financial backing of the Federal Reserve didn’t curb the firm’s appetite for wagering its capital on trading, a formula that fueled Wall Street profit and compensation records in 2007. Second-quarter earnings and revenue also benefited from reduced competition, following the collapse of Bear Stearns Cos. and Lehman Brothers Holdings Inc.

“Our model really never changed,” Goldman Sachs Chief Financial Officer David Viniar said yesterday in an interview. “We’ve said very consistently that our business model remained the same.”

Goldman Sachs, which was the biggest U.S. securities firm before converting to a bank, is the only one of its major Wall Street rivals that hasn’t been transformed by the financial crisis. Lehman Brothers filed for bankruptcy in September, while Bear Stearns Cos. was taken over by JPMorgan Chase & Co. and Merrill Lynch & Co. was sold to Bank of America Corp.

Capital Buffer

Morgan Stanley, which ranked second to Goldman, has said it is scaling back trading risk and principal investments. The firm acquired control of Citigroup Inc.’s Smith Barney brokerage in May to focus on selling financial advice to clients. Analysts predict Morgan Stanley will report a third consecutive quarterly loss next week, after disappointing them with weaker-than-expected trading revenue last quarter.

Goldman Sachs’s value-at-risk, or VaR, has climbed in tandem with the buffer of capital the firm has at its disposal. The bank’s total shareholder equity was $62.8 billion at the end of the second quarter, up from $44.8 billion at the end of the second quarter of 2008, boosted by stock sales in September and again in April.

While the risk-taking has paid off for Goldman Sachs so far, some question whether it could be a perilous example for others to follow.

“Do we want other people trying to emulate what they’re doing, perhaps not with the same skill or resources?” asked Arthur Wilmarth, a professor at George Washington University law school who specializes in issues related to banking. “Regulators ought to be concerned and say ‘Is Goldman making this money with any kind of reasonable prudence?’”

‘Disciplined Fashion’
Equities trading generated a record $3.18 billion of revenue, up 59 percent from the first quarter and 28 percent from a year earlier. Trading in fixed-income, currencies and commodities brought in $6.8 billion, topping last quarter’s record by 4 percent.

“Goldman saw that they were being paid to take risk and they did the appropriate thing,” said Peter Sorrentino, a senior portfolio manager at Huntington Asset Advisors in Cincinnati, which oversees $13.8 billion including Goldman Sachs shares. “If you can generate commensurate return to the risk you’re taking, you do it and you do it in a disciplined fashion.”

Equities trading revenue benefited from a gain in many of the major world markets, with the Standard & Poor’s 500 Index rising 12.6 percent between March 27, the end of Goldman’s fiscal first-quarter, and June 26, the end of its second quarter. The U.K.’s FTSE-100 climbed 8.8 percent during the same period and the Hang Seng index jumped 31.7 percent. Rates, Currencies

Rates, Currencies

“The interesting thing about the equity market is it probably has the most correlation of any market between the direction of prices and how we do,” Viniar said yesterday in a conference call with analysts. “It is the one market where you tend to see more activity when the market’s going up because people are more confident. They feel better. They do more.”

Goldman Sachs trimmed its average daily risk on interest rates to $205 million from $218 million in the first quarter. The figure was still up from $144 million in the second quarter of last year.

Viniar, who turns 54 next week, told analysts that interest-rate revenue in the second quarter was “strong but down from a record first quarter as client activity declined and spreads narrowed modestly.”

Revenue from trading currencies rose in the quarter from the previous period on higher trading volumes, Viniar said. The firm’s average daily risk in currencies rose to $39 million from $38 million in the first quarter, the company reported.

Value-at-risk to commodity prices was unchanged from the first quarter at $40 million and Viniar said revenue in that segment was down from the first quarter because customer activity diminished.

Goldman Sachs reduced the assets on its balance sheet to $890 billion on June 26 from $925 billion at the end of March. The company’s leverage, the ratio of the bank’s assets to shareholder equity, dropped in the quarter to 14.2 from 14.6 at the end of March.

“I do not expect our balance sheet to stay this low,” Viniar said. “We’re in an environment where all of the opportunities are in very, very liquid items, so things move off our balance sheet very quickly.”

So-called Level 3 assets, those that are hardest to value and trade, fell to about $54 billion, or 6 percent of total assets, from $59 billion in the first quarter, or 6.4 percent.

The table below shows Goldman Sachs’s average daily value-at-risk and shareholder equity, from 2007 to present, according to company reports.
<table>
<thead>
<tr>
<th>Quarter End</th>
<th>Value-at-Risk (Daily Average)</th>
<th>Shareholder Equity (at Quarter End)</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 26, 2009</td>
<td>$245 million</td>
<td>$62.81 billion</td>
</tr>
<tr>
<td>March 27, 2009</td>
<td>$240 million</td>
<td>$63.55 billion</td>
</tr>
<tr>
<td>Nov. 28, 2008</td>
<td>$197 million</td>
<td>$64.37 billion</td>
</tr>
<tr>
<td>Aug. 29, 2008</td>
<td>$181 million</td>
<td>$45.60 billion</td>
</tr>
<tr>
<td>May 30, 2008</td>
<td>$184 million</td>
<td>$44.82 billion</td>
</tr>
<tr>
<td>Feb. 29, 2008</td>
<td>$157 million</td>
<td>$42.63 billion</td>
</tr>
<tr>
<td>Nov. 30, 2007</td>
<td>$151 million</td>
<td>$42.80 billion</td>
</tr>
<tr>
<td>Aug. 31, 2007</td>
<td>$139 million</td>
<td>$39.12 billion</td>
</tr>
<tr>
<td>May 25, 2007</td>
<td>$133 million</td>
<td>$38.46 billion</td>
</tr>
<tr>
<td>Feb. 23, 2007</td>
<td>$127 million</td>
<td>$36.90 billion</td>
</tr>
</tbody>
</table>

To contact the reporter on this story: Christine Harper in New York at charper@bloomberg.net.
Solving Large-Scale Complex Capital Budgeting Problems

1. When capital is rationed in one or more periods, no longer should we merely rank projects according to their NPV and just continue to select them in order until the budgets are exhausted.
2. Objective is to find the combination of projects that will maximize net present value while not violating any relevant constraints.
3. Due to disparities in the original costs of projects under consideration, which may find that several projects with smaller original costs have a greater combined NPV than one larger project.

**Mathematical Programming: Zero-One Optimization**

Maximize NPV = 14x₁ + 17x₂ + 40x₃

Subject To:
Expenditures Yr 1  12x₁ + 54x₂ + 30x₃ ≤ 75
Expenditures Yr 2  03x₁ + 07x₂ + 35x₃ ≤ 38
Mutually Exclusive  01x₁ + 00x₂ + 01x₃ ≤ 1
Contingency        01x₁ - 01x₂ + 00x₃ ≤ 0

x₁ ≤ 1
x₂ ≤ 1
x₃ ≤ 1

These constraints are embedded into the solution technique.
1. Produce the canonical form of the zero-one capital investment problem.

2. Solve for the optimal solution where 1 indicates accept and 0 reject the associated project.
Policy Constraints – How To:

This section assumes that the student has had an introduction to constrained optimization (e.g., linear programming, integer programming, etc.). The constraint system presented below assumes a zero-one constrained optimization problem (a special case of integer programming). This means that each decision variable can take a solution value of either zero (0) or one (1). A value of one indicates an accept decision whereas a value of zero should be interpreted as a rejection.

Mutually Exclusive: \( X_i + X_k \leq 1 \)

Contingent (a) \( X_i - X_k = 0; \) if project \( k \) is accepted, then project \( i \) must be accepted.

Contingent (b) \( X_i + X_k \geq 1 \) at least one project must be accepted.

Contingent (c) \((i + k)X_j \leq X_i + X_k \)

- or - \( 2X_j \leq X_i + X_k \)

- or - \( -X_i - X_k - 2X_j \geq 0 \)

accept project \( j \) only if \( i \) and \( k \) are accepted.

Time Delay: a) define a new decision variable (insert column), \( X_j \)

b) \( X_i + X_j \leq 1 \)
Conditional combinations:

Projects $i$ and $k$ and projects $j$ and $p$ can be combined into complementary or composite projects wherein total cash flows will be reduced by 10% and NPV increased by 12% compared with the total of the separate projects.

a) define two new variables: $g$ and $h$

\[
X_i + X_k + X_g \leq 1 \\
X_j + X_p + X_h \leq 1 \\
X_g + X_h \geq 1 
\]

b) adjust objective function and constraint coefficients for projects $g$ and $h$ as required.

The Optimal Capital Investment Budget

Whew… it has been a long trip, but we are now ready to summarize our approach to the capital investment problem.

1. In a perfect capital market characterized by certainty, it is possible to rely upon simple ranking techniques.
2. In this world, using the WinORS output from the capital investment analysis it is possible to observe the cumulative effects of adopting projects in order of their rank (see the cumulative benefits section with a focus on NPV or Adjusted NPV).
3. However, when the problem is large and complex the projects are generally not independent. In this case the decision-maker may choose to characterize the dependencies (including mutually exclusive projects) by the use of the constrained integer programming method.
4. The optimal solution produced by the zero-one programming method may be compared directly to the optimal budget under simple ranking.
5. Under the ranking approach the optimal budget is comprised of the projects with positive NPVs down to the last positive NPV project accepted by firm management and the value of the combined (portfolio) NPV is simply the sum of the NPVs.
6. Under the zero-one approach, the projects with a solution value of one (1) are in the optimal budget and the combined (portfolio) NPV is the objective function value of the optimization solution.
7. The combination of the portfolio in 5 and 6 above are not likely to be the same.
8. In addition to the problem of solving the large-scale complex capital investment problem, we have to decide where to place the management emphasis.
9. The answer is simple: effective control of current and future cash flow (firm profitability from operations).
10. This begins, in part, with a study of the micro-economic theory of demand. It is here where the firm interacts with its customers to create the top-most line on the P/L statement (gross sales, revenues, etc.). This is a study of price*quantity.
**ALGEBRIAC SUBSTITUTIONS**

(1) \[ V = \text{Do} + \frac{D_1}{(1+i)} \]

(2) \[ \text{Do} = X_0 - I_0 + F_0 \]

(3) \[ D_1 = X_1 - (1 + i)F_0 \]

\[
\begin{align*}
V &= X_0 - I_0 + F_0 + \frac{X_1 - (1 + i)F_0}{(1+i)} \\
V &= X_0 - I_0 + F_0 + \frac{X_1}{(1+i)} - \frac{(1 + i)F_0}{(1+i)} \\
V &= X_0 - I_0 + \frac{X_1}{(1+i)} - F_0 \\
V &= X_0 - I_0 + \frac{X_1}{(1+i)}
\end{align*}
\]

**How Net Cash Flows are Measured:**

\[ \text{NCF} = \Delta \text{NIAT} + \Delta D \]

where, \[ \Delta \text{NIAT} = (\Delta R - \Delta C - \Delta D)(1 - t) \]

\[ \text{NCF} = (\Delta R - \Delta C - \Delta D)(1 - t) + \Delta D \]

Where R is firm revenue, C is cost and D is depreciation.

*End of lecture.*